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# Semi-analytic calculation of the monopole order parameter in QCD

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The monopole order parameter of QCD is computed in terms of gauge invariant field strength correlators. Both quantities are partially known from numerical simulations on the lattice. A new insight results on the structure of the confining vacuum.

## 1. Introduction

The mechanism of confinement by dual supercoductivity of the $QCD$ vacuum [1][2][3] is confirmed by numerical simulations on a lattice. Chromoelectric flux tubes connecting $q-\bar{q}$ pairs produced by dual Meissner effect are indeed observed in lattice configurations with the expected form of the electric and magnetic fields[4]. An extensive analysis has been performed by exploring the vacuum by means of an order parameter $\langle\mu\rangle$ [5,6,7,8,9] which is the vacuum expectation value of a magnetically charged operator $\mu$. In the confined phase $\langle\mu\rangle \neq 0$ , which implies Higgs breaking of the magnetic U(1) symmetry, in the deconfined phase $\langle\mu\rangle = 0$ and the magnetic charge is superselected. In SU(N) gauge theory there exist $N-1$ magnetic charges and $N-1$ independent operators $\mu^a, (a = 1,..,N-1)$, which create monopoles of the species $a$[10]. They can be written

$$\mu^a(\vec{x},t) = e^{\frac{iq}{2g}\int d^3y \vec{b}_\perp(\vec{x}-\vec{y})Tr(\Phi^a\vec{E})(\vec{y},t)} \quad (1)$$

$q$ is an integer $\vec{b}_\perp$ is the field of a Dirac monopole with $\vec{\nabla}\vec{b}_\perp = 0$ and $\vec{\nabla}\wedge\vec{b}_\perp = \frac{\vec{r}}{r^3} + Dirac-string$.

$$\Phi^a(x) \equiv U(x,y)\Phi^a_d U^\dagger(x,y) \quad (2)$$

with $U(x,y)$ an arbitrary gauge transformation. We will take for U a parallel transport to $x$ from a reference point $y$ along a path $C$ .

$\langle\mu^a\rangle$ is gauge invariant. In Eq(2)

$$\Phi^a_d \equiv diag(\overbrace{\frac{N-a}{N},..,\frac{N-a}{N}}^{<--a-->}\ ,\ \overbrace{-\frac{a}{N},..,-\frac{a}{N}}^{<-(N-a)>}) \quad (3)$$

In the gauge in which $\Phi^a = \Phi^a_d$ (Abelian Projection)

$$\mu^a(\vec{x},t) = e^{\frac{iq}{2g}\int d^3y \vec{b}_\perp(\vec{x}-\vec{y})\vec{E}^a_\perp(\vec{y},t)} \quad (4)$$

where $\vec{E}^a_\perp$ is the transverse chromoelectric field along the color direction $T^a$

$$T^a = diag(0,...0,\overset{a}{1}\ ,\ \overset{a+1}{-1},0,..0) \quad (5)$$

$\vec{E^a}_\perp$ is the conjugate momentum to $\vec{A^a}_\perp$. Hence

$$\mu^a(\vec{x},t)|\vec{A}^a_\perp(\vec{y},t)\rangle = |\vec{A}^a_\perp(\vec{y},t) + \frac{q}{2g}\vec{b}_\perp(\vec{x}-\vec{y})\rangle \quad (6)$$

$\mu^a(\vec{x},t)$ creates a Dirac monopole at $(\vec{x},t)$ in the residual gauge symmetry after abelian projection. It proves convenient to use instead of $\langle\mu^a\rangle$ the susceptibility

$$\rho^a \equiv \frac{\partial}{\partial\beta} ln\langle\mu^a\rangle \quad (7)$$

Here $\beta$ is the usual variable of the lattice formulation $\beta \equiv \frac{2N}{g^2}$. Since at $\beta = 0$ $\mu^a = 1$,

$$\langle\mu^a\rangle = e^{\int_0^\beta d\beta \rho^a(\beta)} \quad (8)$$

$\rho^a$ has been measured by lattice simulation for various gauge theories:compact $U(1)$ [7],$SU(2)$ [8] , $SU(3)$ [9] and $N_f = 2$ $QCD$ [11] . In all these systems $\rho^a \to finite$ in the confined phase in the thermodynamical limit $V \equiv L_s^3 \to \infty$. By use of Eq(8) this means $\langle\mu^a\rangle \neq 0$ .

In the deconfined phase $T > T_c$

$$\rho^a \approx -|c|L_s + c' \text{ or } \langle\mu^a\rangle = 0 \quad (9)$$

or, again by Eq(8) $\langle\mu^a\rangle = 0$ .

In the critical region $T \approx T_c$ the scaling law holds

$$\frac{\rho^a}{L_s^{\frac{1}{\nu}}} \approx f(\tau L_s^{\frac{1}{\nu}}) \qquad (10)$$

Here $\tau \equiv 1 - \frac{T}{T_c}$ , $\nu$ is the critical index of the correlation length of the order parameter. $\rho^a$ is independent of the choice of the abelian projection [12] [13] [14]. Expanding the exponential which defines $\langle\mu^a\rangle$ one has

$$\langle\mu^a\rangle = \Sigma_0^\infty (\frac{iq}{2N})^n \frac{1}{n!} \int d^3y_1..d^3y_n b^{i_1}(\vec{x}-\vec{y}_1).. \\ b^{i_n}(\vec{x}-\vec{y}_n)\langle(\Phi^a.\vec{E})_{i_1,\vec{y}_1}(\Phi^a.\vec{E})_{i_n,\vec{y}_n}\rangle \qquad (11)$$

The notation is $(\Phi^a.\vec{E})_{i,x} \equiv Tr[\Phi^a(x).E^i(x)$.

The $vev's$ in Eq(11) are gauge invariant field strength correlators.

We shall identify these correlators with those of the Stochastic Vacuum approach to QCD [20] [21] [22].

## 2. Cluster Expansion of $\langle\mu^a\rangle$

In $QCD$ all observables can be expressed in terms of gauge invariant field strength correlators. The basic idea of the stochastic vacuum approach is to perform a systematic cluster expansion of the correlators and truncate it typically retaining only the clusters up to order two. Since the one point cluster $\langle(\Phi^a.\vec{E})\rangle = 0$, only two point clusters will survive the truncation, and hence only even terms in the expansion Eq(11) , which will be approximated as products of two point correlators $\Phi^a_{i_1,i_2}(\vec{y}_1 - \vec{y}_2) = \langle(\Phi^a.\vec{E})_{i_1,\vec{y}_1}(\Phi^a.\vec{E})_{i_2,\vec{y}_2}\rangle$. There is a combinatorial factor $(2n-1)!!$ for the term of order $2n$ . Eq(11) becomes

$$\langle\mu^a\rangle = e^{-\frac{q^2}{8g^2}\int d^3y_1 \int d^3y_2 \Phi^a_{i_1,i_2}(\vec{y}_1-\vec{y}_2) b_\perp^{i_1}(\vec{y}_1) b_\perp^{i_2}(\vec{y}_2)} \qquad (12)$$

or, since $\beta = \frac{2N}{g^2}$

$$\rho^a = -\frac{q^2}{16N}\frac{\partial}{\partial\beta}[\beta \int d^3y_1 \int d^3y_2 \\ \Phi^a_{i_1,i_2}(\vec{y}_1-\vec{y}_2) b_\perp^{i_1}(\vec{y}_1) b_\perp^{i_2}(\vec{y}_2)] \qquad (13)$$

We shall identify $\Phi^a_{i_1,i_2}$ with the two point correlators defined with a straight line parallel transport. These correlators are measured on the lattice [17][18][19] , and are used as an input in stochastic $QCD$. For those correlators $\langle E^a E^b\rangle = \delta^{ab}\Phi$ , so that $\rho^a$ is independent on $a$ . This is also the case in lattice determinations of $\rho^a$ [9].

The the cluster expansion is generically expected to work at large distances, and in the study of confinement we are looking for infrared properties . Anyhow a direct check of it can be obtained by looking at the dependence of $\rho^a$ on $q$. The truncated $\rho^a$ is proportional to $q^2$ : higher correlators would introduce terms proportional to higher powers of $q$. Old data [8] [15] seem to agree with $q^2$ but a systematic study of this dependence will be done.

## 3. The Field Correlators

A general parametrization of field strength correlators dictated by invariance arguments[20] [21] is

$$\Phi^{ab}_{\mu_1,\nu_1,\mu_2\nu_2}(z_1 - z_2) \equiv \frac{1}{N}\langle Tr F^a_{\mu_1\nu_1}(z_1) \\ V(z_1,z_2) F_{\mu_2,\nu_2}(z_2) V^\dagger(z_1,z_2)\rangle \qquad (14)$$

$$\Phi^{ab}_{\mu_1,\nu_1,\mu_2\nu_2}(z_1 - z_2) = \delta^{ab} \\ (D(z_1-z_2)[\delta_{\mu_1\mu_2}\delta_{\nu_1\nu_2} - \delta_{\mu_1\nu_2}\delta_{\nu_1\mu_2}] \\ +\frac{1}{2}\frac{\partial}{\partial z_{\mu_1}}[D_1(z_1-z_2)(z_{\mu_2}\delta_{\nu_1\nu_2} - z_{\nu_2}\delta_{\nu_1\mu_2}] + \\ \frac{1}{2}\frac{\partial}{\partial z_{\nu_1}}[D(z_1-z_2)(z_{\nu_2}\delta_{\mu_1\mu_2} - z_{\mu_2}\delta_{\mu_1\nu_2}]) \qquad (15)$$

At $T \neq 0$ the electric field correlators do not coincide with the magnetic ones, and there are four form factors, $D_E, D_{1E}, D_H, D_{1H}$ .

For correlators of electric fields $E_{i_1}$ , $E_{i_1}$ Eq(15) gives

$$\Phi^{ab}_{i_1 i_2} = \delta^{ab}[\delta_{i_1 i_2}(D_E + \frac{1}{2}D_{1E}) + \frac{\partial}{\partial i_1..}] \qquad (16)$$

In the convolution with $\vec{b}_\perp$ the derivative terms give 0. For the same reason

$\delta_{i_1 i_2} \to \delta_{i_1 i_2} - \frac{k_{i_1}k_{i_2}}{k^2}$

Going to the Fourier transform we get for $\rho^a$ Eq(13)

$$\rho^a = -\frac{q^2}{16}\frac{\partial}{\partial\beta}[\beta \int \frac{d^3k}{(2\pi)^3} b_\perp^{i_1}(\vec{k}) b_\perp^{i_2}(-\vec{k})$$

$$D_E(k^2)\frac{1}{k^2}(k^2\delta_{i_1 i_2} - k_{i_1}k_{i_2})] \tag{17}$$

Here $\bar{D}_E(k^2)$ is the Fourier transform of $(D_E + \frac{1}{2}D_{1E})$.

Since

$$(k^2\delta_{ij} - k_{i1}k_{i2})b_\perp^{i_1}(\vec{k})b_\perp^{i_2}(-\vec{k}) = |\vec{H}(\vec{k})|^2 \tag{18}$$

we can use the explicit form of $\vec{H}(\vec{k})$

$$\vec{H}(\vec{k}) = \vec{k} \wedge \vec{b}_\perp(\vec{k}) \tag{19}$$

with $\vec{n}$ the direction of the Dirac string ( we shall call it $z$), and get

$$|\vec{H}(\vec{k})|^2 = -\frac{1}{k^2} + \frac{1}{k_z^2} \tag{20}$$

For $\rho^a$ we then have

$$\rho^a = \frac{q^2}{16}\frac{\partial}{\partial\beta}[\beta\int\frac{d^3k}{(2\pi)^3}(\frac{1}{k^2} - \frac{1}{k_z^2})f(k^2)] \tag{21}$$

where $f(k^2) \equiv \frac{1}{k^2}D_E(k^2)$.

Identifying our correlators with those of the stochastic model simply explains that $\rho^a$ is independent both on $a$ and on the abelian projection. At large $\beta$(deconfined phase) $f(k^2)$ can be approximated by first order perturbation theory

$$f(k^2) = \frac{1}{2Nk} \tag{22}$$

The only dependence on $\beta$ is the explicit factor in Eq(21) so that

$$\rho^a = \frac{q^2}{16N}\int\frac{d^3k}{(2\pi)^3}\frac{1}{k}(\frac{1}{k^2} - \frac{1}{k_z^2}) \tag{23}$$

The integral is easily computed with UV cut-off $\frac{1}{a}$ ($a$ the lattice spacing) and IR cut-off $\frac{1}{L_s a}$ ($L_s$ the spacial size of the lattice). The result is

$$\rho^a = \frac{q^2}{16N}\frac{1}{(2\pi)^2}[-\sqrt(2)L_s + 2ln(L_s) + const] \tag{24}$$

By comparison with Eq(8) this means that $\langle\mu^a\rangle = 0$ in the thermodynamical limit $L_s \to \infty$. The correlators have been measured on the lattice both at $T = 0$ and at $T \neq 0$. In the range of distances $.1fm \le x \le 1fm$ they are well parametrized by a form [17]

$$D_E = Ae^{-\frac{x}{\lambda_b}} + \frac{b}{x^4}e^{-\frac{x}{\lambda_a}} \tag{25}$$

with $\lambda_a \approx 2\lambda_b$ and $\lambda_b \approx .3fm$ $A$ and $b$ are independent on $\beta$ within statistical errors in the range from $T = 0$ up to $T \approx .95T_c$ . Approaching further $T_c$ $A$ rapidly decreases to zero.[19].

In fact the parametrization Eq(25) cannot be valid at shorter distances, where the operator product expansion and the non-existence of condensates of dimension less than 4 require that

$$D_E \approx \frac{b}{2}[\frac{1}{(x+ie)^4} + \frac{1}{(x-ie)^4}] + c + dx^2 \tag{26}$$

The prescription on the singularity is the same as in perturbation theory. At larger distances a stronger infrared cut-off at some distance $\Lambda$, must exist, since colored particles cannot propagate at infinite distance. This feature needs a further numerical investigation of correlators at large distance on the lattice.

Up to $T \approx .95T_c$ the only dependence on $\beta$ in Eq(21) is again the explicit factor so that the result coincides with Eq(24) except that the lattice size $L_s$ is replaced by the infrared cut-off $\Lambda$

$$\rho^a = \frac{q^2}{16}\frac{1}{(2\pi)^2}[-\sqrt(2)\Lambda + 2ln(\Lambda) + const] \tag{27}$$

The integral of the exponential term of Eq(25) is included in the constant. This expression gives a finite value of $\rho^a$ for $T < T_c$ independent of the volume and hence $\langle\mu^a\rangle \neq 0$ .This means dual superconductivity for any finite value of the UV cut-off $a$. However in the continuum limit $a \to 0$ $\rho^a$ diverges so that $\langle\mu^a\rangle$ needs a renormalization. This is similar to what happens for the Polyakov line [23]. Existing Lattice data support this statement [See Fig(2) of ref.[8]] ,but we plan to do a more systematic investigation of this issue. By approaching the critical temperature both the IR cut-off $\Lambda$ and the coefficient $A$ in Eq(25) strongly depend on $\beta$ :$\Lambda$ diverges and $A$ tends to zero. A more detailed calculation, which will be reported elsewhere, gives

$$\rho^a = \frac{q^2}{16N}\frac{\partial}{\partial\beta}(\beta[\frac{1}{(2\pi)^2}(-\sqrt(2)\frac{\Lambda}{a} + 2\ln(\frac{\Lambda}{a}) + const) + 2N\lambda_b^4 A_E(1 + \frac{\Lambda}{\pi\lambda_b}] \tag{28}$$

Numerical determinations of the dependence of $A_E$ on of the temperature around $T_c$ exist in the literature[19]. Not much is known about the behavior of $\Lambda$ . Further study is needed to understand how the scaling law of $\rho^a$ Eq(10) described in Sect 1 comes out of Eq(28).

## 4. Conclusions

We close with a few remarks.

Checking the dependence $\rho^a \propto q^2$ is a test of the Stochastic Vacuum model.

The independence of $\rho^a$ on $a$ and on the abelian projection are also an important test of it.

The existence of confinement depends on a strong infrared cut-off of the field correlators. This can be directly checked on lattice..

$\rho^a$ diverges in the continuum limit, but provides a good description of confinement at any fixed value of the UV cut-off.

An interesting interplay emerges of confinement with infrared properties of gauge invariant field strength correlators.

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